

# Knightian Robustness of Single-Parameter Domains

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## Abstract

We consider players that have very limited knowledge about their own valuations. Specifically, the only information that a *Knightian player*  $i$  has about the profile of true valuations,  $\theta^*$ , consists of a set of distributions, from one of which  $\theta_i^*$  has been drawn.

We prove a “robustness” theorem for Knightian players in single-parameter domains: every mechanism that is weakly dominant-strategy truthful for classical players continues to be well-behaved for Knightian players that choose undominated strategies.

# 1 Introduction

In [CMZ14] we motivate the problem of mechanism design for Knightian players, and prove that (1) dominant-strategy mechanisms for single-good and multi-unit auctions cannot provide good social-welfare efficiency, but (2) the second-price and Vickrey mechanisms deliver good social-welfare performance, for these two settings, in undominated strategies.

In this report, we prove a “robustness” theorem for single-parameter domains. Namely, consider a mechanism  $M$  for a single-parameter domain and suppose that  $M$ , when players have perfect information about their own valuations, is weakly dominant-strategy truthful. Now consider the same mechanism  $M$ , but with Knightian players that, not having any dominant strategy to play, choose to play undominated strategies. We prove that the set of undominated strategies is well-behaved, in the sense that these strategies do not deviate from the players’ approximate information about his own valuation.

## 2 Model

In a classical single-parameter domain, there is a set  $\mathcal{A}$ , the set of all possible allocations; for each player  $i$  there exists a publicly known subset  $\mathcal{S}_i \subseteq \mathcal{A}$ ; and the set of possible valuations for player  $i$ ,  $\Theta_i$ , consists of all functions mapping  $\mathcal{A}$  to the reals, subject to the following constraints: for each  $\theta_i \in \Theta_i$ ,

- (1)  $\theta_i(x) = 0 \quad \forall x \notin \mathcal{S}_i$  and
- (2)  $\theta_i(x) = \theta_i(y) \quad \forall x, y \in \mathcal{S}_i$ .

We denote the true valuation of player  $i$  by  $\theta_i^*$ .

(The term “single-parameter” derives from the fact that each  $\theta_i \in \Theta_i$  coincides with a single number:  $i$ ’s value for, say, the lexicographically first element of  $\mathcal{S}_i$ . The term “classical” emphasizes that each player knows exactly his own true valuation.)

The set of possible outcomes is  $\Omega \stackrel{\text{def}}{=} \mathcal{A} \times \mathbb{R}_{\geq 0}^n$ . If  $(A, P) \in \Omega$ , we refer  $P_i$  as the price charged to player  $i$ . We assume quasi-linear utilities. That is, the utility function  $U_i$  of a player  $i$  maps a valuation  $\theta_i$  and an outcome  $\omega = (A, P)$  to  $U_i(\theta_i, \omega) \stackrel{\text{def}}{=} \theta_i(A) - P_i$ .

If  $\omega$  is a distribution over outcomes, we also denote by  $U_i(\theta_i, \omega)$  the expected utility of player  $i$ .

Single-parameter domains are general enough to include several settings of interest: in particular, provision of a public good<sup>1</sup> [Cla71], bilateral trades [MS83], and buying a path in a network [NR01].

## 2.1 Knightian Valuation Uncertainty

In our model, a player  $i$ 's sole information about  $\theta^*$  consists of  $\mathcal{K}_i$ , a set of distributions over  $\Theta_i$ , from one of which  $\theta_i^*$  has been drawn. (The true valuations are uncorrelated.) That is,  $\mathcal{K}_i$  is  $i$ 's sole (and private) information about his own true valuation  $\theta_i^*$ . Furthermore, for every opponent  $j$ ,  $i$  has no information (or beliefs) about  $\theta_j^*$  or  $\mathcal{K}_j$ .

Given that all he cares about is his expected (quasi-linear) utility, a player  $i$  may ‘collapse’ each distribution  $D_i \in \mathcal{K}_i$  to its expectation  $\mathbb{E}_{\theta_i \sim D_i}[\theta_i]$ .<sup>2</sup> Therefore, for single-parameter domains, a *mathematically equivalent* formulation of the Knightian valuation model is the following:

**Definition 2.1** (Knightian valuation model). *For each player  $i$ ,  $i$ 's sole information about  $\theta^*$  is a set  $K_i$ , the candidate (valuation) set of  $i$ , such that  $\theta_i^* \in K_i \subset \Theta_i$ .*

*We refer to an element of  $K_i$  as a candidate valuation.*

In Knightian valuation model, a mechanism's performance will of course depend on the inaccuracy of the players' candidate sets, which we measure as follows.

**Definition 2.2.** *Let  $K_i^\perp \stackrel{\text{def}}{=} \inf K_i$  and  $K_i^\top \stackrel{\text{def}}{=} \sup K_i$ .*

*The candidate set  $K_i$  of a player  $i$  is (at most)  $\delta$ -approximate if  $K_i^\top - K_i^\perp \leq \delta$ .*

*A single-parameter domain is (at most)  $\delta$ -approximate if each  $K_i$  is  $\delta$ -approximate.*

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<sup>1</sup>Indeed, in the provision of a public good,  $\mathcal{A}$  has just two elements,  $a$  (i.e., the good is provided), which different players may value differently, and  $b$  (i.e., the good is not provided), which all players value 0.

<sup>2</sup>Whatever the auction mechanism used, this equivalence holds for any auction where each  $\Theta_i$  is a *convex* set. In particular, this includes unrestricted combinatorial auctions of  $m$  distinct goods.

## 2.2 Social Welfare, Mechanisms, and Knightian Dominance

**Social welfare.** The social welfare of an allocation  $A \in \mathcal{A}$ ,  $\text{SW}(A)$ , is defined to be  $\sum_i \theta_i^*(A)$ ; and the maximum social welfare, MSW, is defined to be  $\max_{A \in \mathcal{A}} \text{SW}(A)$ . (That is, SW and MSW continue to be defined relative to the players' true valuations  $\theta_i^*$ , whether or not the players know them exactly.)

More generally, the social welfare of an allocation  $A$  relative to a valuation profile  $\theta$ ,  $\text{SW}(\theta, A)$ , is  $\sum_i \theta_i(A)$ ; and the maximum social welfare relative to  $\theta$ ,  $\text{MSW}(\theta)$ , is  $\max_{A \in \mathcal{A}} \text{SW}(\theta, A)$ . Thus,  $\text{SW}(A) = \text{SW}(\theta^*, A)$  and  $\text{MSW} = \text{MSW}(\theta^*)$ .

**General mechanisms and strategies.** A mechanism  $M$  specifies, for each player  $i$ , a set  $S_i$ . We interchangeably refer to each member of  $S_i$  as a pure *strategy/action/report* of  $i$ , and similarly, a member of  $\Delta(S_i)$  a mixed strategy/action/report of  $i$ .

After each player  $i$ , simultaneously with his opponents, reports a strategy  $s_i$  in  $S_i$ ,  $M$  maps the reported strategy profile  $s$  to an outcome  $M(s) \in \Omega$ .

If  $M$  is probabilistic, then  $M(s) \in \Delta(\Omega)$ . Thus, as per our notation,  $U_i(\theta_i, M(s)) \stackrel{\text{def}}{=} \mathbb{E}_{\omega \sim M(s)}[U_i(\theta_i, \omega)]$  for each player  $i$ .

Note that  $S_i = \Theta_i$  for the direct mechanisms in the classical setting, but may be arbitrary in general.

**Knightian undominated strategies.** Given a mechanism  $M$ , a pure strategy  $s_i$  of a player  $i$  with a candidate set  $K_i$  is (*weakly*) *undominated*, in symbols  $s_i \in \text{UD}_i(K_i)$ , if  $i$  does not have another (possibly mixed) strategy  $\sigma_i$  such that

- (1)  $\forall \theta_i \in K_i \forall s_{-i} \in S_{-i} \quad \mathbb{E}U_i(\theta_i, M(\sigma_i, s_{-i})) \geq U_i(\theta_i, M(s_i, s_{-i}))$ , and
- (2)  $\exists \theta_i \in K_i \exists s_{-i} \in S_{-i} \quad \mathbb{E}U_i(\theta_i, M(\sigma_i, s_{-i})) > U_i(\theta_i, M(s_i, s_{-i}))$ .

If  $K$  is a product or a profile of candidate sets, that is, if  $K = (K_1, \dots, K_n)$  or  $K = K_1 \times \dots \times K_n$ , then  $\text{UD}(K) \stackrel{\text{def}}{=} \text{UD}_1(K_1) \times \dots \times \text{UD}_n(K_n)$ .

Note that the above notion of an undominated strategy is a natural extension of its classical counterpart, but other extensions are possible.

**Weakly dominant-strategy truthfulness in classical settings.** Finally, let us recall what it means for a mechanism  $M$  to be weakly dominant-strategy truthful

(weakly DST) when every player  $i$  knows  $\theta_i^*$  exactly. Namely, for each player  $i$ :

- (0)  $S_i = \Theta_i$
- (1)  $\forall v_i \in \Theta_i \forall v'_i \in \Theta_i \forall v_{-i} \in \Theta_{-i} \quad U_i(v_i, M(v_i, v_{-i})) \geq U_i(v_i, M(v'_i, v_{-i}))$
- (2)  $\forall v_i \in \Theta_i \forall v'_i \in \Theta_i \setminus \{v_i\} \exists v_{-i} \in \Theta_{-i} \quad U_i(v_i, M(v_i, v_{-i})) > U_i(v_i, M(v'_i, v_{-i}))$  .

(For comparison, the notion of a DST mechanism omits the last condition above.)

### 3 Result

We prove the Knightian robustness of many mechanisms at once as follows.

**Theorem 1.** *Let  $M$  be a weakly dominant-strategy truthful mechanism for classical single-parameter domains. Then, in this domain with Knightian valuation uncertainty, for every player  $i$ ,  $\text{UD}(K_i) \subseteq [K_i^\perp, K_i^\top]$ .*

**Discussion.** The above theorem implies that the behavior of (weakly dominant-strategy truthful) mechanisms in a  $\delta$ -approximate single-parameter domains gracefully degrades with  $\delta$ . In particular, it implies that, when applied to the provision of a public good in the presence of  $n$  Knightian players, the VCG mechanism guarantees, in undominated strategies, a social welfare  $\geq \text{MSW} - 2n\delta$ . As another example, when applied to buying paths in a network, the VCG mechanism guarantees a social welfare  $\geq \text{MSW} - 2m\delta$ , where  $m$  is the number of edges in the network. Finally, we note that the proof of Theorem 1 easily extends to imply an analogous result for the VCG mechanism for *single-minded combinatorial auctions*, which are not quite single-parameter domains.<sup>3</sup>

More generally, Theorem 1 implies that, for all weakly dominant-strategy mechanisms  $M$  (which include those of [Cla71, MS83, NR01])

*‘the outcome  $M(v)$  is sufficiently good*

*whenever  $\max_i |v_i - \theta_i^*|$  is sufficiently small for all  $i$  and  $\theta_i^* \in K_i$ ’.*

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<sup>3</sup>In such an auction, there are  $m$  distinct goods, and each player  $i$  values, positively and for the same amount  $\theta_i^*$ , only the supersets of a given subset  $\mathcal{S}_i$  of the goods. This auction is not single-parameter because  $\mathcal{S}_i$  is *private*, that is, known solely to  $i$ . Accordingly,  $i$ ’s true valuation can be fully described only by the number  $\theta_i^*$  and the subset  $\mathcal{S}_i$ . The VCG mechanism for single-minded auctions ensures, in undominated strategies, a social welfare that is at least  $\text{MSW} - 2 \min\{n, m\}\delta$ .

**Proof.** The theorem is obvious when  $K_i = \{\theta_i^*\}$  is a singleton: since reporting the truth is a weakly dominant strategy, it dominates all other strategies so that  $\text{UD}(K_i) = \{\theta_i^*\}$  must also be a singleton. For the rest of the proof we assume that  $K_i$  has at least two distinct valuations.

We begin by recalling the following fact about dominant-strategy truthful mechanisms in single-parameter domains where each player perfectly knows his own true valuation [AT01]:

Let  $M$  be a mechanism for a single-parameter domain, and let  $f_i(v) \in [0, 1]$  be the probability that the allocation chosen by  $M$ , under strategy profile  $v$ , is in player  $i$ 's set  $\mathcal{S}_i$ . Then,  $M$  is dominant-strategy truthful if and only if (a)  $f$  is monotonically non-decreasing, i.e.,  $f_i(v_i, v_{-i}) \leq f_i(v'_i, v_{-i})$  whenever  $v_i \leq v'_i$ , and (b) player  $i$ 's expected price on input  $v$ , denoted by  $p_i(v)$ , equals to  $v_i \cdot f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(z, v_{-i}) dz$ .

Having recalled the above fact, we now prove that, for any Knightian player  $i$  with candidate set  $K_i = [K_i^\perp, K_i^\top]$ ,

$$v_i \in \text{UD}_i(K_i) \implies v_i \in [K_i^\perp, K_i^\top].$$

Let  $v_i^\perp \stackrel{\text{def}}{=} K_i^\perp$  and  $v_i^\top \stackrel{\text{def}}{=} K_i^\top$ , and consider any strategy  $v_i \in \text{UD}_i(K_i)$ . If  $v_i \in K_i = [v_i^\perp, v_i^\top]$  then we are done. Otherwise, suppose that  $v_i < v_i^\perp$ . (The other case,  $v_i > v_i^\top$ , can be shown analogously.)

We first claim that, for player  $i$ , reporting  $v_i^\perp$  is no worse than reporting  $v_i$ . Indeed, fixing any (pure) strategy sup-profile  $v_{-i}$  for the other players and any possible true valuation  $\theta_i \in K_i$ , and letting  $v^\perp = (v_i^\perp, v_{-i})$  and  $v = (v_i, v_{-i})$ , we compute that

$$\begin{aligned} & \mathbb{E}[U_i(\theta_i, M(v^\perp))] - \mathbb{E}[U_i(\theta_i, M(v))] \\ &= (f_i(v^\perp) - f_i(v)) \cdot \theta_i - (p_i(v^\perp) - p_i(v)) \\ &= (f_i(v^\perp) - f_i(v)) \cdot \theta_i - \left( v_i^\perp \cdot f_i(v^\perp) - \int_0^{v_i^\perp} f_i(z, v_{-i}) dz - v_i \cdot f_i(v) + \int_0^{v_i} f_i(z, v_{-i}) dz \right) \\ &= (f_i(v^\perp) - f_i(v)) \cdot (\theta_i - v_i^\perp) + \int_{v_i}^{v_i^\perp} (f_i(z, v_{-i}) - f_i(v)) dz. \end{aligned}$$

Now note that  $\theta_i \in K_i$  implies that  $\theta_i - v_i^\perp = \theta_i - K_i^\perp \geq 0$ . Moreover, by the monotonicity of  $f$ , whenever  $z \geq v_i$ , it holds that  $f_i(z, v_{-i}) \geq f_i(v)$ . Therefore we

deduce that the above difference is greater than or equal to zero. We conclude that reporting  $v_i^\perp$  is no worse than reporting  $v_i$ .

Next there are two subcases. If  $\mathbb{E}[U_i(\theta_i, M(v^\perp))] - \mathbb{E}[U_i(\theta_i, M(v))]$  equals to zero for all  $\theta_i \in K_i$  and for all  $v_{-i}$ , then, using the fact that  $K_i$  has at least two distinct valuations, we conclude that for  $i$ , the allocation probability and (expected) price in outcomes  $M(v_i, v_{-i})$  and  $M(v_i^\perp, v_{-i})$  are the same, independent of  $v_{-i}$ . This contradicts the fact that  $M$  is weakly dominant-strategy truthful in the classical setting, since  $U_i(v_i, M(v_i, v_{-i}))$  must be strictly greater than  $U_i(v_i, M(v_i^\perp, v_{-i}))$  at least for some  $v_{-i}$ .

Otherwise, if there exist some  $\theta_i^*$  and some  $v_{-i}^*$  that make the difference  $\mathbb{E}[U_i(\theta_i, M(v^\perp))] - \mathbb{E}[U_i(\theta_i, M(v))]$  non-zero, it must follow that the difference is strictly positive. For such  $\theta_i^*$  and  $v_{-i}^*$ , reporting  $v_i^\perp$  is therefore strictly better than reporting  $v_i$ , so by definition  $v_i^\perp$  weakly dominates  $v_i$  for player  $i$ , leading to a contradiction to  $v_i \in \text{UD}_i(K_i)$ .

This concludes the proof of Theorem 1. ■

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